The Refolding Tree Viewpoint of Iterated (HD) Systems

Abstract

Iterated systems of the form

$x_{t+1} = HDx_t$

where H is a normalized Walsh–Hadamard transform and D is a diagonal sign matrix, exhibit dynamical behavior that is often richer than expected from conventional linear orthogonal systems. While the system is globally linear and norm-preserving, repeated iteration produces complex transient interference structures, metastable oscillations, and globally distributed phase mixing.

This note develops a “refolding tree” interpretation of iterated (HD) dynamics. In this viewpoint, each iteration recursively branches state information through Hadamard expansion while simultaneously refolding all branches back into a globally coupled interference field. The resulting dynamics resemble recursive scattering on a recombining binary tree rather than conventional modal oscillation.

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1. The Basic System

Consider:

$$A = HD$$

where:

- $H \in \mathbb{R}^{N \times N}$ is the normalized Walsh–Hadamard transform,
- D is diagonal with entries ± 1,
- $N = 2^n$.

Since both H and D are orthogonal:

$$A^T A = I.$$

Thus:

- all eigenvalues lie on the unit circle,
- total energy is conserved,
- dynamics are globally rotational.

Despite this apparent simplicity, iterated evolution:

$x_t = A^t x_0$

often develops highly structured internal interference patterns.

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2. Recursive Expansion Structure

The key observation is that the Hadamard transform possesses an intrinsic recursive branching structure:

$$H_{2n} = \frac{1}{\sqrt{2}} \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

Each recursion level:

1. duplicates information into paired channels,

2. applies opposing sign structure,
3. recursively repeats this splitting.

Operationally, one application of H can be interpreted as:

- recursive binary expansion,
- distributed sign modulation,
- multiscale interference redistribution.

When combined with diagonal reflections D, each iteration alternates between:

- local phase/sign modification,
- global recursive redistribution.

This creates the basis for the “refolding tree” interpretation.

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3. The Refolding Tree Interpretation

A useful mental model is:

> Each iteration expands the state into an exponentially branching interference tree, then globally refolds the branches back together.

The distinction from ordinary tree propagation is crucial.

In a conventional branching process:

- branches separate over time.

In iterated HD:

- branches separate,
- but are repeatedly recombined into the full state space.

Thus the geometry is not a pure tree, but rather:

> a recursively expanding and refolding interference tree.

Each state component at time t receives contributions from exponentially many interference paths generated across prior iterations.

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4. Path Expansion Perspective

After t iterations:

$A^t = (HD)^t$.

Expanding this recursively generates a combinatorial superposition of sign-dependent propagation paths.

Each coordinate of x_t becomes:

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$$x_i(t) = \sum_{\text{paths } p} \sigma(p) x_{j(p)}(0),$$

where:

- p indexes branching histories,
- $\sigma(p) \in \{\pm 1\}$ represents accumulated sign products,
- $j(p)$ maps the path back to an initial coordinate.

The number of effective pathways grows exponentially with iteration depth.

This produces:

- large-scale cancellation structures,
- transient coherent packets,
- energy migration between interference families,
- metastable oscillatory behavior.

The resulting dynamics resemble:

- wave scattering,
- recursive diffraction,
- combinatorial propagation systems,
  more than classical Fourier oscillator systems.

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5. Why Fourier Analysis Often Performs Poorly

Fourier methods assume:

- stable sinusoidal modes,
- localized spectral content,
- stationary harmonic decomposition.

Iterated HD systems violate these assumptions operationally.

Although exact eigenmodes exist, they are typically:

- highly delocalized,
- globally distributed,
- strongly mixed in coordinate space.

For random or generic D:

- local observables contain contributions from many rotational modes simultaneously,
- modal phases continually interfere under recursive redistribution.

Thus FFT measurements often exhibit:

- broad spectra,
- unstable peaks,
- apparent spectral diffusion,
- pseudo-random oscillatory envelopes.

The Fourier basis is not aligned with the recursive branching geometry of the system.

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6. Refolding as the Source of Complexity

The essential phenomenon is not expansion alone, but repeated refolding.

Pure expansion would produce separable branching trajectories.

Pure orthogonal rotation would produce stable modal oscillation.

Iterated HD instead performs:

1. recursive branch generation,
2. global interference recombination,
3. redistribution into new branch structures.

This repeated folding operation causes:

- long-range coupling,
- distributed phase entanglement,
- multiscale interference cascades.

The system remains linear, yet observable structures behave similarly to nonlinear emergent systems.

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7. Relation to Hypercube Geometry

Walsh functions are characters of the Boolean group:

$$(\mathbb{Z}_2)^n.$$

Thus H) evolution may also be interpreted as propagation on the Boolean hypercube.

From this viewpoint:

- recursive branching corresponds to hypercube traversal,
- Hadamard mixing corresponds to global character redistribution,
- diagonal sign modulation acts as local phase reflection.

The “tree” observed in low-dimensional projections may therefore be a shadow of higher-dimensional hypercube propagation.

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8. Metastable Oscillatory Structures

One experimentally noticeable property is the appearance of transient oscillatory structures after perturbations to D.

For example:

- flipping one sign in D,

- modifying a small subset of diagonal entries,
- introducing localized phase defects,

often produces:

- slowly evolving interference envelopes,
- apparent migration of energy between oscillation families,
- long transient relaxation phases.

In the refolding-tree viewpoint, such perturbations alter:

- branching phase compatibility,
- cancellation structure,
- recombination geometry.

The resulting transients resemble:

- scattering resonances,
- interference defects,
- topological phase perturbations.

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9. Linear Systems with Emergent Observational Complexity

A central implication of this viewpoint is that:

- linearity alone does not imply observational simplicity.

The complexity of iterated HD systems arises from:

- recursive global mixing,
- combinatorial pathway growth,
- repeated interference refolding.

Thus:

- the underlying operator remains linear and orthogonal,
- yet observable behavior may resemble nonlinear emergent dynamics.

This places iterated HD systems in an unusual regime:

- structurally linear,

- phenomenologically complex.

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10. Open Directions

The refolding-tree viewpoint suggests several possible research directions:

- characterization of interference branch families,
- invariant structures under recursive refolding,
- localization phenomena in structured D,
- Krylov-subspace descriptions of metastable oscillations,
- entropy growth of branch interference patterns,
- relations to quantum walk dynamics,
- hypercube scattering analogies,
- compressed representations of distributed rotational structure.

There may also exist connections to:

- holographic representations,
- associative memory systems,
- recursive coding structures,
- random projection dynamics,
- distributed phase computation.

Conclusion

Iterated HD systems may be more naturally interpreted not as ordinary collections of coupled oscillators, but as recursively expanding and refolding interference structures.

The Hadamard transform generates binary branching propagation, while diagonal sign reflections continuously perturb interference geometry. Repeated iteration creates a globally coupled recombining tree whose observable behavior can exhibit surprisingly rich emergent structure despite strict underlying linearity.

The “refolding tree” viewpoint provides an intuitive geometric interpretation for:

- metastable oscillations,
- distributed phase behavior,
- transient interference packets,

- apparent nonlinear phenomenology,
- and the difficulty of extracting simple modal descriptions using Fourier analysis alone.